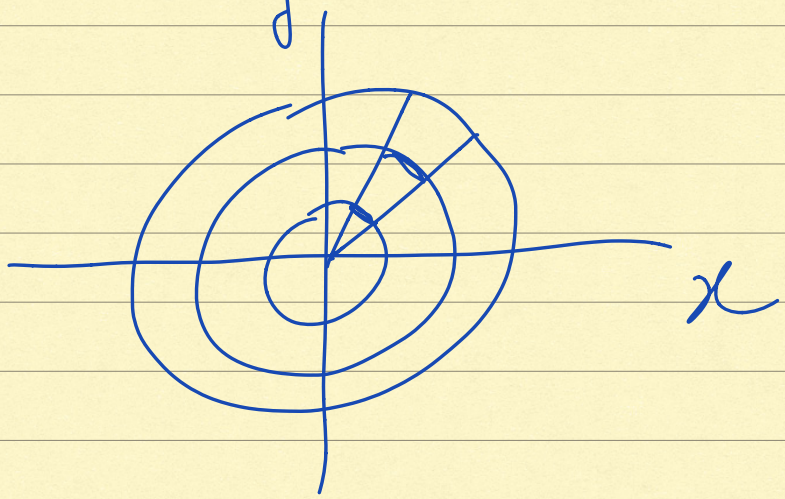
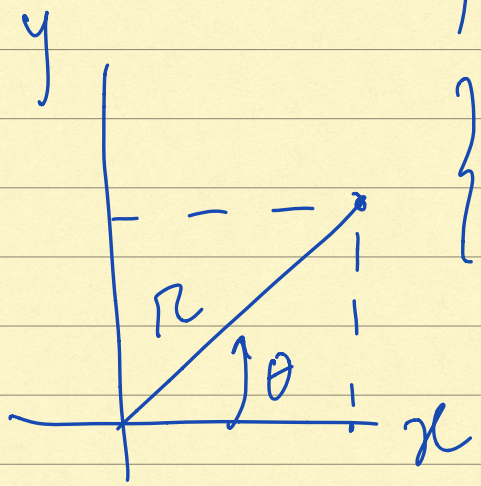
Mudança de Variáveis:

$$X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$





$$\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

$$(x,y) = g(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta)$$

$$g :]0,+\infty[\times]0,2\pi[\longrightarrow \mathbb{R}^2, C^1$$

r $\quad$ θ

$$g(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta) = (x,y)$$

invertível $\equiv$ injectiva

$$(r,\theta) = g^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \operatorname{arctan}\frac{y}{x}\right)$$

$$\det Dg(r,\theta) = r \neq 0$$

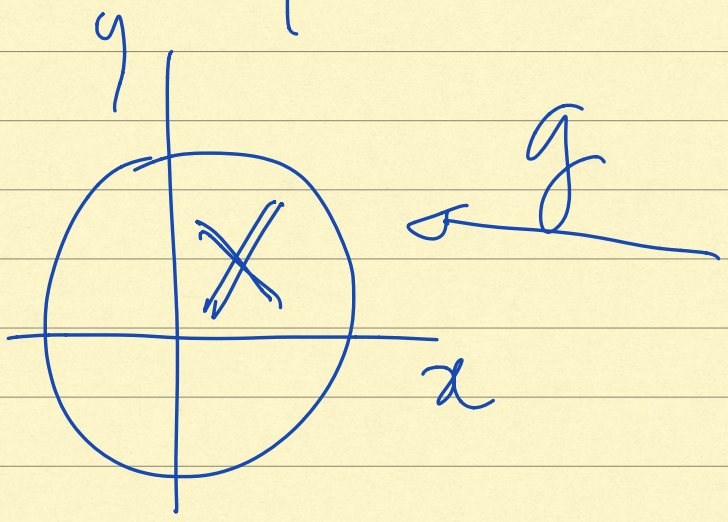
Definição: Se $g : \mathbb{R}^4 \to \mathbb{R}^4$, C^1, injectiva e $\det Dg(x) \neq 0, \forall x$ diz-se que g é uma MUDANÇA DE VARIÁVEIS.

$$\iint_X dx\,dy = \iint_T r\,dr\,d\theta$$

X T

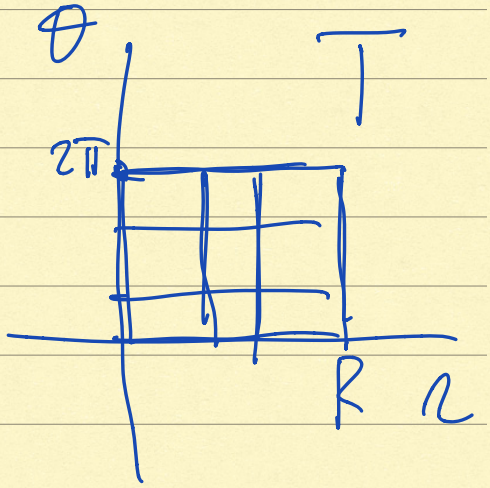
$$X: \boxed{x^2+y^2 \le R^2}$$

$$T: \begin{cases} 0 < \theta < 2\pi \\ 0 < r < R \end{cases}$$



Complicado

"dados"



Simples

$$\int_X f(x)\,dx = \int_T f(g(t))\,|\det Dg(t)|\,dt$$

$x = g(t)$

$g : \mathbb{R}^n \to \mathbb{R}^n$, mudança de variáveis

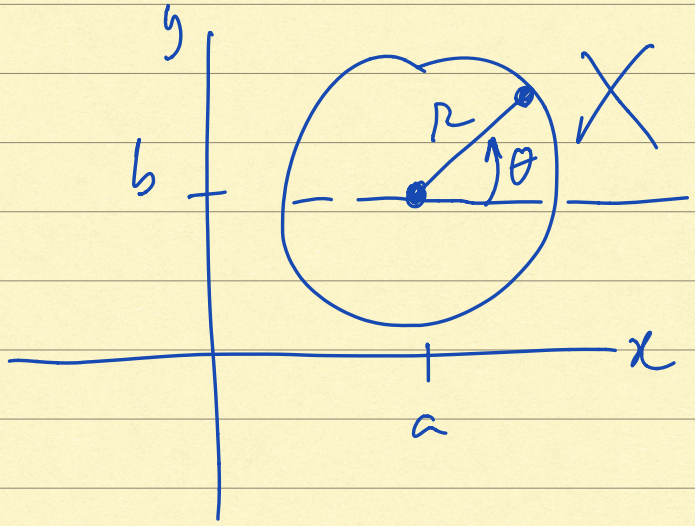
"complicado" → "simples"

$\mathbb{R}$:
$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\,|g'(t)|\,dt$$

$g : \mathbb{R} \to \mathbb{R}$

X, f → $\int_X f$ → mudar

Ex: $(x-a)^2+(y-b)^2 \leq R^2$



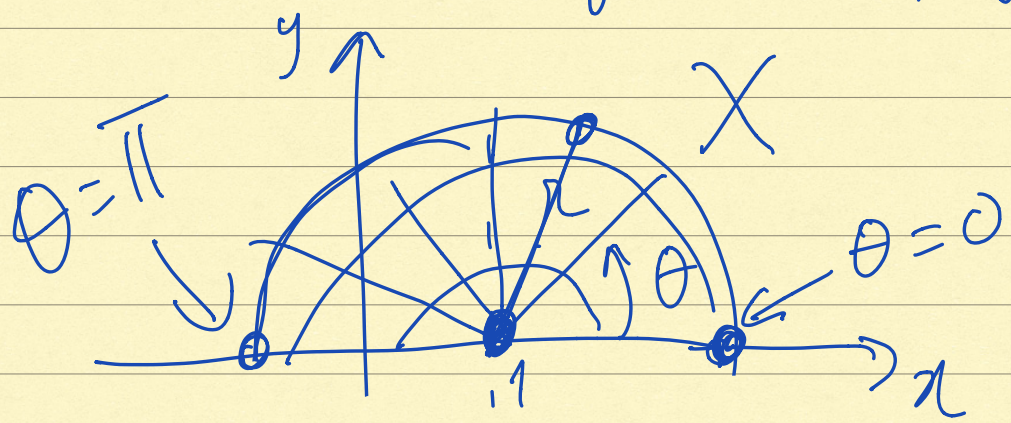
$$\begin{cases} x-a = r\cos\theta \\ y-b = r\operatorname{sen}\theta \end{cases} \quad \begin{cases} x = a+r\cos\theta \\ y = b+r\operatorname{sen}\theta \end{cases}$$

$$g(r,\theta) = (a+r\cos\theta,\ b+r\operatorname{sen}\theta) = (x,y)$$

$$\det Dg(r,\theta) = r.$$

$$\iint_X f = \int_0^{2\pi}\left(\int_0^R r\, f(g(r,\theta))\,dr\right)d\theta$$

Ex. X: $(x-1)^2 + y^2 \le 4 \; ; \; y \ge 0$



$r = \sqrt{(x-1)^2 + y^2}$

$f(x,y) = y$

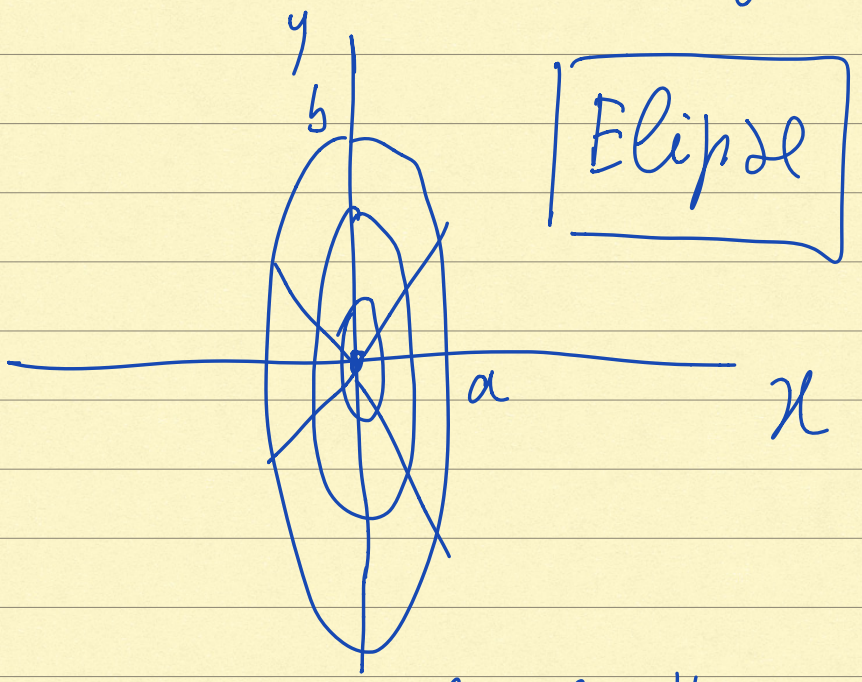
$$\begin{cases} x - 1 = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

$$\begin{matrix} 0 \le \theta \le \pi \\ 0 \le r \le 2 \end{matrix} \; T$$

$$\iint_X f = \int_0^{\pi} \left(\int_0^2 r(r\operatorname{sen}\theta)\, dr \right) d\theta$$

$$= \int_0^{\pi} \operatorname{sen}\theta \, \frac{8}{3} \, d\theta = \frac{8}{3}\left(1+1\right) = \frac{16}{3}$$

Exemplo: $X: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$



Elipse

"Soma de quadrados" → cos, sen

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

$$\begin{cases} \frac{x}{a} = r\cos\theta \\ \frac{y}{b} = r\,\text{sen}\,\theta \end{cases} \qquad \begin{cases} x = ar\cos\theta \\ y = br\,\text{sen}\,\theta \end{cases}$$

$$g(r,\theta) = (ar\cos\theta,\ br\operatorname{sen}\theta)$$

$$\begin{cases} 0 < \theta < 2\pi \\ 0 < r < 1 \end{cases} \qquad \boxed{\det Dg(r,\theta) = abr}$$

Exercício.

Área da elipse X.

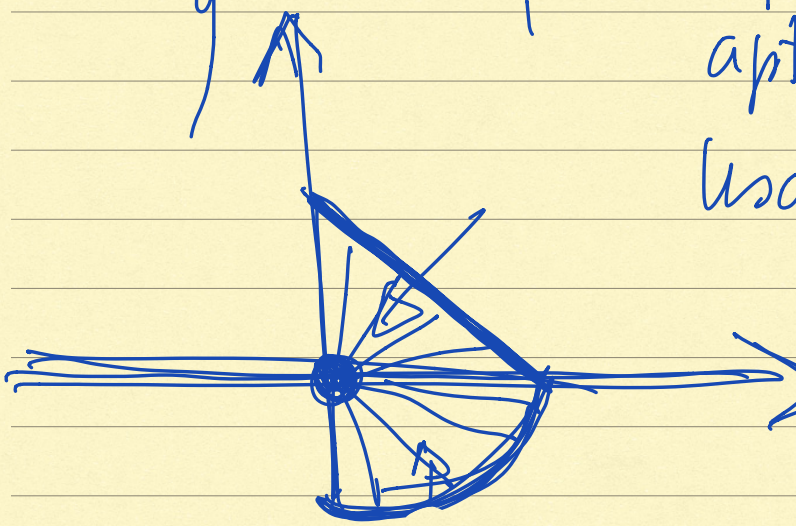
$$\operatorname{vol}_2(X) = \iint_X 1 = \int_0^{2\pi}\left(\int_0^1 abr\,dr\right)d\theta$$

$$= \int_0^{2\pi} \frac{ab}{2}\,d\theta = \boxed{\pi ab}$$

Se $a = b \longrightarrow$ círculo de raio a.

$$\boxed{\pi a^2}$$

figuras não aptas para usar coord. polares

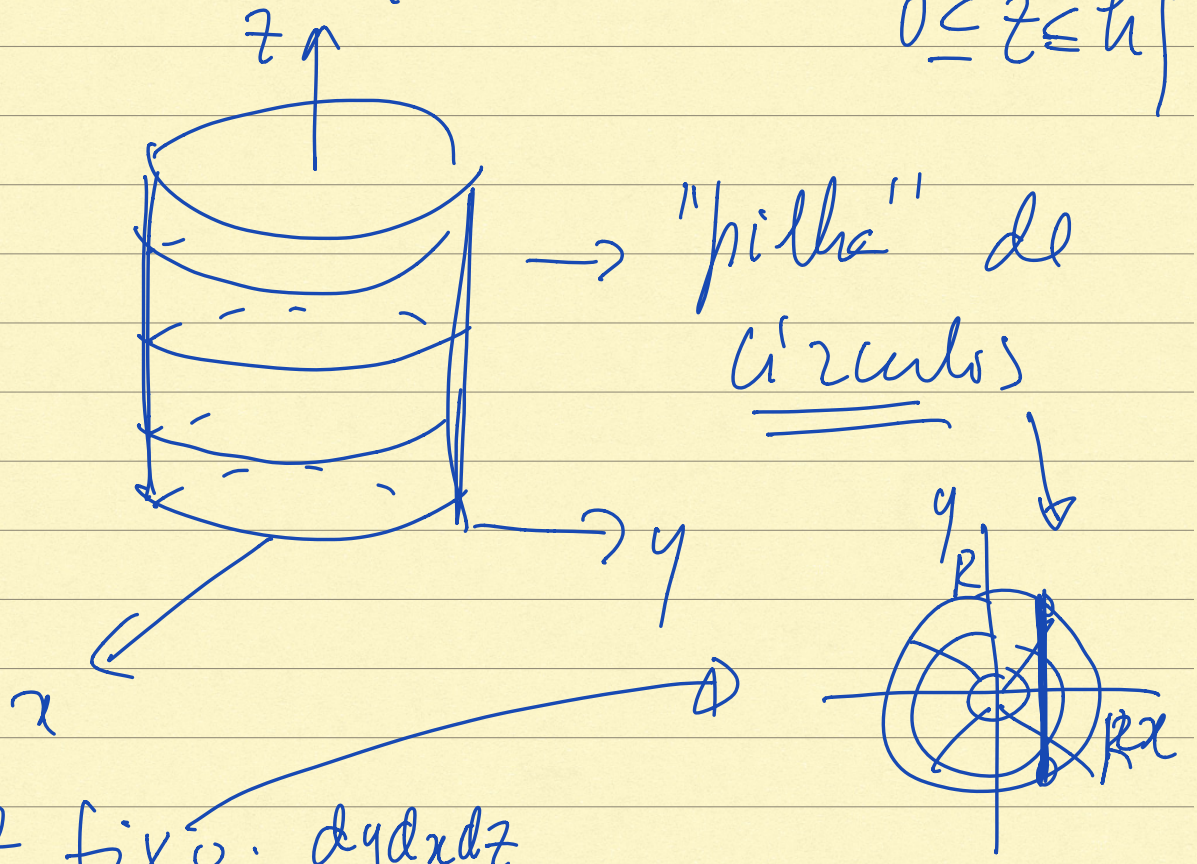


"Soma de dois quadrados"

= circunferências

→ cos(·), sen(·)

$\mathbb{R}^3$: $X = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq R^2$
$0 \leq z \leq h\}$



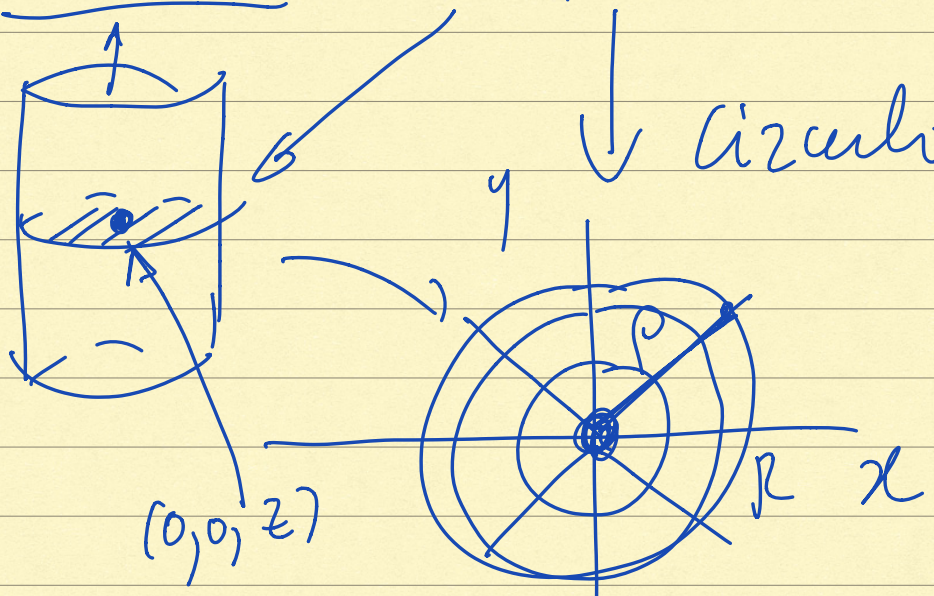
→ "pilha" de círculos

z fixo: $dy\,dx\,dz$

$$\int_0^h \left(\int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \right) dx \right) dz$$

difícil!!!

Mudar: z fixo



↓ círculo → polares ρ, θ

$x = \rho \cos\theta$

$y = \rho \operatorname{sen}\theta$

ρ = distância ao ponto $(0,0,z)$ (distância ao eixo Oz)

$$\boxed{\rho = \sqrt{x^2 + y^2}}$$

$(\rho, \theta, z) \longleftrightarrow (x, y, z)$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \\ z = z \end{cases}$$

$$g(\rho,\theta,z) = (\rho\cos\theta, \rho\operatorname{sen}\theta, z)$$

↑ eixo de simetria.

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \\ z = z \end{cases}$$

$$\boxed{\det Dg(\rho,\theta,z) = \rho \quad \text{exercício.}}$$

$$\operatorname{Vol}_3(X) = \int_0^{2\pi}\left(\int_0^R\left(\int_0^h \rho\, dz\right) d\rho\right) d\theta$$

$$= \cdots = \underbrace{\pi R^2}\times\underbrace{h}$$

$x^2 + y^2 \leq R^2$; $0 \leq z \leq h$

ρ^2

(ρ, θ, z)